Sinosoids and the Frequency Domain

Sinosoids and other periodic functions

As we will see, each periodic signal (imagine for example a sawtooth-wave) can be seen as a sum of weighted (scaled) and time-shifted sine-waves with different frequencies. A generic sinosoidal signal can be described by the formula:

$$x(t) = \sin(t) \tag{1}$$

The sine function is a periodic function and we will denote it's period (it's interval of repitition) by T. If we measure the time t in seconds, this signal has a period of $T = 2\pi = 6.28...$ seconds and therefore a frequency of $\frac{1}{2\pi}Hz$. By multiplying the argument t of the sine with this period 2π, we obtain a signal which has a new period of 1 second and also a frequency of $1Hz$. If we now additionally multiply the argument of the sine by some arbitrary chosen factor (say f), we get a sine-wave with frequency f. Thus a sine-wave at an arbitrary frequency f is given by:

$$x(t) = \sin(2\pi f t) = \sin(\omega t) \tag{2}$$

where the definition of the radian frequency $\omega := 2\pi f$ was introduced for convenience (ω is a lowercase greek omega). Now that we have defined a sinoisoidal signal with an arbitrary frequency, we need two additional parameters to 1.) shift the sinosoid along the time-axis and 2.) scale the sinosoid to the desired amplitude. A time shift is achieved by adding a constant φ (a lowercase greek phi) to the argument of the sine - this constant φ is also known as the phase-shift of the sine. The scaling is achieved by a simple multiplication of the whole signal by some factor A. Thus, the general description of a sine-wave with radian frequency ω, phase-shift φ and Amplitude A is given by:
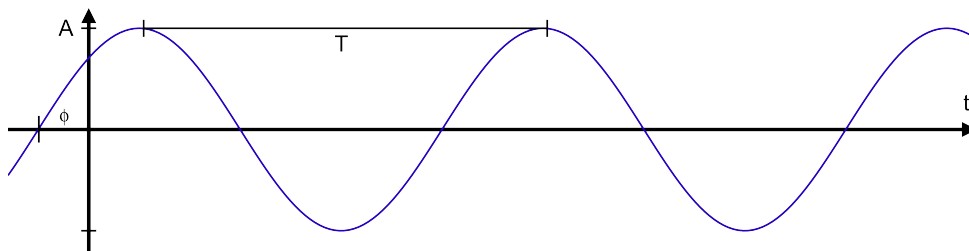
$$x(t) = A\sin(\omega t + \varphi) \tag{3}$$

Figure 1: A generic sinosoid with its parameters amplitude A, period $T = \frac{1}{f} = \frac{2\pi}{\omega}$ and phase φ

To see how to construct an arbitrary waveform from such sinosoids, we'll stick to the example of the sawtooth-wave: A sawtooth-wave of radian frequency ω_1 can be constructed by first taking a sinosoid of frequency ω_1 with amplitude A_1, adding a second sinosoid with a frequency twice as high ($\omega_2 = 2\omega_1$, that is) and with half the amplitude $A_2 = \frac{A_1}{2}$, then adding a third sinosoid with radian frequency $\omega_3 = 3\omega_1$ and Amplitude $A_3 = \frac{A_3}{3}$ and so forth (we don't need the phase parameter for the sawtooth, i.e. can set it to zero). The sum is formally expressed as:

$$x(t) = \sum_{n=1}^{N} A_n \sin(\omega_n t + \varphi_n) \tag{4}$$

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where $\omega_n = n \cdot \omega_1$ is the radian frequency of the nth sinosoid, $A_n = \frac{A_1}{n}$ is its amplitude and $\varphi_n = 0$ is its phase (which could have been dropped here because it is zero for the sawtooth, but has been included for completeness). The upper summation index can be some finite positive number or infinity (where only in the latter case we obtain the exact sawtooth-wave and an approximation for finite N). This process of summing sinosoids is illustrated in figure 2 The important point to note is: **the more terms we add,**
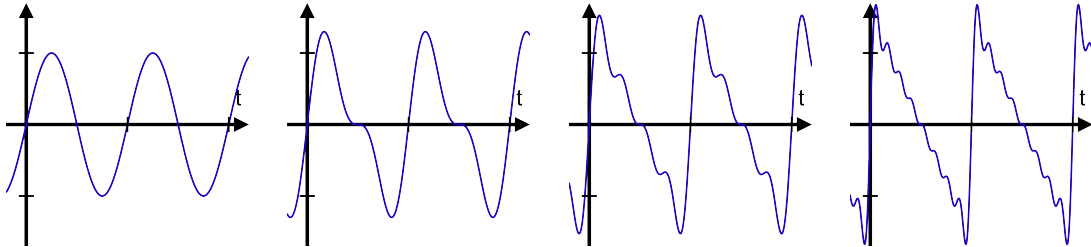

Figure 2: Composing a sawtooth from sinosoids - 1, 2, 4 and 8 harmonics

the closer our sum of sinosoids becomes to a sawtooth-wave and if we take the limit to add an infinite number of terms, the sum converges exactly to the sawtooth-wave (there are some mathematical subtleties about this convergence, but we shall not concern ourselves with these here). Equation 4 works for any waveform, just that the expressions for the amplitudes A_n and phases φ_n will be different. Note that **in this summation only (radian) frequencies occur, which are integer multiples of the lowest frequency** ω_1 (which is the reciprocal of the period) - this is generally true for periodic waveforms and **we refer to this lowest (radian) frequency as the fundamental (radian) frequency**. The process of adding together sine-waves to obtain an arbitrary waveform is also known as additive synthesis (for obvious reasons) or - more mathematically - Fourier synthesis in honour to the french mathematician Jean Baptiste Joseph Fourier who found all this out. Conversely to Fourier synthesis, we can break down any periodic waveform into such a sum of scaled and possibly shifted sine-waves with frequencies that are multiples of the fundamental frequency. The process of finding the Amplitudes A_n and phases φ_n for those components is then called Fourier analysis (the radian frequencies ω_n are known in advance).

Why are sinosoids elementary?

Sinosoids are not the only kind of waveform from which we can construct an arbitrary other waveform by adding scaled versions of various frequencies. It is also possible to construct any waveform by adding scaled and shifted rectangular pulse-functions, for example. So what is so special about this particular waveform, that it was chosen to serve as - so to speak - the most atomic one? A deep mathematical reason has to do with one of the most important building blocks of signal processing algorithms - filters. A filter accepts a signal at its input and produces another signal at its output, where the output-signal is the filtered version of the input signal. In general, this output-signal will look completely different from the input-signal when displayed on an oscilloscope - however, it turns out that sinosoids are the only kind of signal for which the output will always look similar to the input. More precisely and technically: **for sinosoidal input-signals, all a filter can do is to change its amplitude and introduce some time-delay, but the shape of the waveform will stay exactly the same**. In more perceptual or musical terms this means: a filter can change only the volume of a sinosoid (and delay it a bit), but the timbre will stay the same. ...and a DSP-engineer would probably put it like: sinosoids are the

eigenfunctions of linear time-invariant systems. These 3 statements all mean basically the same thing in different terminology. **This very fact makes it possible to characterize filters in terms of their frequency response** (the term 'frequency-response' is mostly used synonymous to the technically more correct term 'magnitude-response'). The frequency-response shows, how sinosoids at different frequencies will be attenuated or boosted. Sinosoids are also meaningful elementary signals in a perceptual sense, because our hearing system works (to a coarse approximation) very much like a filter bank. Another interesting (and somewhat related) fact about sinosoids is, that a sinosoid of a given frequency, amplitude and phase can be added to another sionosoid of the same frequency (but possibly with a different amplitude and phase) and the result is again a sinosoid at the same frequency, the amplitude and phase of which is determined by the amplitudes and phases of the two component sinosoids.

The spectrum of a signal

We have just seen that periodic signals can be described as sums of sinosoids - however, Fourier theory goes a step further. Not only can we describe periodic signals as sums of sinosoids, but we can also describe nonperiodic signals by means of sinosoids - for that we have to relax the restriction that the frequencies should be integer multiples of some fundamental frequency. If we allow for all frequencies, our frequency-variable ω now becomes a continuous quantity (instead of being restricted to discrete values $\omega_1, \omega_2, \ldots$) and so the amplitude becomes a continuous function of ω, that is: $A = A(\omega)$, and likewise the phase denoted as $\varphi(\omega)$. The sum of equation 4 must therefore be replaced by an integral, so we write for an arbitrary signal $x(t)$:

$$x(t) = \int A(\omega) \sin(\omega t + \varphi(\omega)) \, d\omega \tag{5}$$

and the functions $A(\omega)$ and $\varphi(\omega)$ are called the magnitude-spectrum and phase-spectrum of the signal $x(t)$ respectively.

What's the point of all this?

Now it may appear, that we have made simple things complicated by expressing simple functions such as sawtooth-waves (which is just a linear ramp which repeats over and over) as infinite sums of conceptually more complex functions such as sinosoids. There must be good reasons to do so and indeed there are. First, as already mentioned, filters can most conveniently described by their action on sinosoids - that is: once we know what a filter does to sinosoids of various frequencies, we know everything there is to know about the filter. Second, our perception is mostly based on a frequency domain description of signals as well - intuitively, a signal with lots of high frequency sinosoids in it corresponds to a bright sound whereas a signal with few or no high frequency sinosoids in it is perceived as dull. The simple bass/treble controls which are present on virtually every HiFi-system do nothing else then boost or attenuate low and high frequency sinosoids for example.